Solving Recurrence Relations

- **Definition:** A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:
- $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$,
- Where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- A sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions

- $a_0 = C_0, a_1 = C_1, a_2 = C_2, \ldots, a_{k-1} = C_{k-1}$.

Solving Recurrence Relations

- **Examples:**
- The recurrence relation $P_n = (1.05)P_{n-1}$
- is a linear homogeneous recurrence relation of **degree one**.

- The recurrence relation $f_n = f_{n-1} + f_{n-2}$
- is a linear homogeneous recurrence relation of **degree two**.

- The recurrence relation $a_n = a_{n-5}$
- is a linear homogeneous recurrence relation of **degree five**.

Solving Recurrence Relations

- Basically, when solving such recurrence relations, we try to find solutions of the form $a_n = r^n$, where r is a constant.
- $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if
- $r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_kr^{n-k}$.
- Divide this equation by r^{n-k} and subtract the right-hand side from the left:
- $r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_{k-1}r - c_k = 0$
- This is called the **characteristic equation** of the recurrence relation.

Solving Recurrence Relations

- The solutions of this equation are called the **characteristic roots** of the recurrence relation.

- Let us consider linear homogeneous recurrence relations of **degree two**.

- **Theorem:** Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2.
- Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

- See pp. 321 and 322 for the proof.

Solving Recurrence Relations

- **Example:** What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

- **Solution:** The characteristic equation of the recurrence relation is $r^2 – r – 2 = 0$.
- Its roots are $r = 2$ and $r = -1$.
- Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if:
- $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ for some constants α_1 and α_2.

Solving Recurrence Relations

- Given the equation $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that
- $a_0 = 2 = \alpha_1 + \alpha_2$
- $a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$

- Solving these two equations gives us
$\alpha_1 = 3$ and $\alpha_2 = -1$.

- Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with
- $a_n = 3 \cdot 2^n - (-1)^n$.

Solving Recurrence Relations

- $a_n = r^n$ is a solution of the linear homogeneous recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

- if and only if
- $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$.
- Divide this equation by r^{n-k} and subtract the right-hand side from the left:
- $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$
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Solving Recurrence Relations

- **Example:** Give an explicit formula for the Fibonacci numbers.
- **Solution:** The Fibonacci numbers satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.
- The characteristic equation is $r^2 - r - 1 = 0$.
- Its roots are

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Solving Recurrence Relations

- Therefore, the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

for some constants ?$_1$ and ?$_2$.
We can determine values for these constants so that the sequence meets the conditions

$f_0 = 0$ and $f_1 = 1$:

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$$

Solving Recurrence Relations

•The unique solution to this system of two equations and two variables is

$$\alpha_1 = \frac{1}{\sqrt{5}}\ ,\ \ \alpha_2 = -\frac{1}{\sqrt{5}}$$

So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving Recurrence Relations

- But what happens if the characteristic equation has only one root?
- How can we then match our equation with the initial conditions a_0 and a_1 ?
- **Theorem:** Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for n = 0, 1, 2, …, where α_1 and α_2 are constants.

Solving Recurrence Relations

- **Example:** What is the solution of the recurrence relation $a_n = 6a_{n-1} – 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?
- **Solution:** The only root of $r^2 – 6r + 9 = 0$ is $r_0 = 3$.

Hence, the solution to the recurrence relation is

- $a_n = \alpha_1 3^n + \alpha_2 n3^n$ for some constants α_1 and α_2.
- To match the initial condition, we need
- $a_0 = 1 = \alpha_1$

$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$

- Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$.
- Consequently, the overall solution is given by
- $a_n = 3^n + n3^n$.